TURBULENT BOUNDARY LAYER

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On the basis of a superposition model of turbulent eddies in contact with a solid wall, some new laws regarding the behavior of the turbulent boundary layer are established.

To calculate heat- and mass-transfer processes in technical equipment, it is necessary to know the laws of turbulent boundary-layer flow.

Among the theories establishing turbulent boundary-layer laws, a special place belongs to that of Millionshchikov [1], which has made a significant contribution to the turbulent boundary-layer problem.

Writing Newton's law for a viscous liquid

$$\frac{\tau_W}{\rho} = \nu \frac{\partial u}{\partial y} \tag{1}$$

in the dimensionless form

$$\frac{\partial u^+}{\partial y^+} = 1 \tag{2}$$

and using the integral of Eq. (2)

$$u^+ = y^+, \tag{3}$$

satisfying the condition of liquid contact at a motionless wall, Millionshchikov obtained the equation

$$y^+ \frac{\partial u^+}{\partial y^+} = u^+, \tag{4}$$

which was given the following physical interpretation: "Laminar flow may be represents as the superposition of eddies fluctuating at the wall with dimensionless angular velocity $\partial u^+/\partial y^+$, with a fluctuation velocity at a point y^+ equals to the flow velocity u^+."

Millionshchikov then considered turbulent flow, like laminar flow, in the form of a superposition of eddies, but reaching the boundary surface of a laminar sublayer characterized by the dimensionless thickness δ_0^+.

On the basis of the condition that liquid flow occurs with a minimum of kinetic anergy corresponding to the specified liquid flow related to the flow resistance, Millionshchikov obtained the basic equation of his theory in the form

$$(y^+ - \delta_0^+) \frac{\partial u^+}{\partial y^+} = \frac{1}{\varkappa}, \tag{5}$$

where δ_0^+ and $\varkappa$ are empirical constants.

Comparing Eqs. (5) and (4), it is evident that, despite a well-founded physical model, Millionshchikov did not obtain a generalized equation for the turbulent boundary layer that was free from empirical constants.

Below, it will be shown that a generalized equation not containing any empirical constants can also be obtained for the case of a turbulent boundary layer.

First of all, Eq. (2) will be analyzed.

If the function $l_*(x)$, defined by the expression

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$$l_*(x) = \frac{\nu}{u_*(x)}, \tag{6}$$

is taken as the scale for the variable y, the function $u(x, y)$ is transformed to the form $u'(x, y^+)$. After dividing $u'(x, y^+)$ by the dynamic velocity $u_*(x)$, it is found that

$$u^+(x, y^+) = \frac{u'(x, y^+)}{u_*(x)}. \tag{7}$$

Hence, Eq. (2) is essentially the integral of the more general equation

$$\frac{\partial}{\partial x} \frac{\partial u^+(x, y^+)}{\partial y^+(x, y)} = 0. \tag{8}$$

Solving Eq. (8) gives, taking definite boundary conditions into account, Eq. (2).

On the basis of the analogy with Eq. (4) and the condition in Eq. (8), the basic equation of a turbulent boundary layer is written in the form

$$R \frac{\partial U}{\partial R} = U, \tag{9}$$

where the generalized dimensionless velocity U satisfies the condition

$$\frac{\partial}{\partial x} \frac{\partial U(x, R)}{\partial R(x, y)} = 0. \tag{10}$$

Here R is the generalized dimensionless distance from the wall.

By definition, the function $U(x, R)$ of the argument $R(x, y)$ involving the normalized variable x will be regarded as continuous in the closed interval

$$R_1 \leqslant R \leqslant R_2, \tag{11}$$

if in any segment

$$R_A \leqslant R \leqslant R_B \tag{12}$$

of this interval the following condition is satisfied

$$\frac{\partial}{\partial x} \int_{R_A}^{R_B} U dR = 0, \tag{13}$$

where

$$R_2 \geqslant R_B \geqslant R_A \geqslant R_1 \geqslant 0. \tag{14}$$

It is also assumed that the turbulent boundary layer is a continuum of continually interacting local liquid masses (eddies), and that after a sufficient time interval, each point of the turbulent boundary layer in a reference frame fixed in the solid wall will correspond to an averaged local mass. In this sense, the turbulent boundary layer is, as it were, quantized over the eddies.

Since the reason for the appearance of turbulent eddies in an initially unperturbed liquid is the introduction of a moving solid wall, the initial eddies (eddies of minimal dimension) arise precisely at the wall. Therefore, it is expedient to consider the region of definition of the mean longitudinal velocity in the form

$$0 \leqslant x_1 \leqslant x \leqslant x_2, \; 0 < l_*(x) \leqslant y \leqslant \delta(x). \tag{15}$$

Here $l_*(x)$ is the mean longitudinal dimension (radius) of the eddy forming at the wall; $\delta(x)$ is the boundary-layer thickness.

Suppose that, in the first approximation, $l_*(x)$ coincides with Eq. (6). It will be shown that if $u(x, y)$ with the region of definition in Eq. (15) may be formulated in agreement with the normalized function $U(R)$, then the expressions

$$U = \frac{u^+ - 1}{u_\delta^+ - 1} \tag{16}$$

and

$$R = \frac{\ln y^+}{\ln \delta^+} \tag{17}$$

are the desired functions.

In fact, the condition in Eq. (13) means that the function U(r) is unique. The two-dimensional region of definition in Eq. (15) must then be converted into a curve R(x, y), and the boundary curves $l_*(x)$ and $\delta(x)$ must correspond to two values of R.

For clarity, the functions $u(x_1, y)$ and $u(x_2, y)$ are shown in Fig. 1a, where y is regarded as the argument and x as a parameter. The shaded region corresponds to possible values of u(x, y).

The function $y^+(x, y)$ is obtained by dividing y by the scale $l_*(x)$ (Fig. 1b).

Next, the expression

$$U'(x,\ y^+) = \frac{u'(x,\ y^+) - u_*(x)}{u_\delta(x) - u_*(x)} = \frac{u^+(x,\ y^+) - 1}{u_\delta^+(x) - 1}$$

is introduced (Fig. 1c). At first glance, it appears that for the points $(\delta_1^+, 1)$ and $(\delta_2^+, 1)$ to coincide, $R(y^+)$, must be chosen as the ratio $R = f(y^+)/f(\delta^+)$, where $f(y^+)$ is an arbitrary function. However, this affects the convergence point (1, 0) in Fig. 1c, since the function $R = f(1)/f(\delta^+)$ depends on the value of δ^+.

This does not occur if the function $\ln y^+$ is taken as $f(y^+)$. In this case, Eqs. (16) and (17) are obtained (Fig. 1d).

But it does not follow from the foregoing that each value of R corresponds to only one value of U, since Eqs. (16) and (17) are necessary but not sufficient for Eq. (13) to hold over the whole of the segment $[R_1, R_2]$.

Dividing U by R, the function Ψ, which depends on the values of u^+ and y^+, is introduced

$$\frac{U}{R} = \frac{\Psi_\delta}{\Psi}, \tag{18}$$

where

$$\Psi_\delta = \frac{\ln \delta^+}{u_\delta^+ - 1}, \tag{19}$$

$$\Psi = \frac{\ln y^+}{u^+ - 1}. \tag{20}$$

The following result may be obtained from Eq. (18)

$$\frac{\partial U}{\partial R} = \frac{\Psi_\delta}{\Psi}\left(1 - \frac{R}{\Psi}\frac{\partial \Psi}{\partial R}\right). \tag{21}$$

Substituting Eq. (21) into Eq. (10) yields

$$\frac{\partial}{\partial x}\frac{\Psi_\delta}{\Psi}\left(1 - \frac{R}{\Psi}\frac{\partial \Psi}{\partial R}\right) = 0. \tag{22}$$

To satisfy Eq. (13) simultaneously, the relation $R_2 = 1$ is assumed in Eq. (11).

Equation (22) has two particular solutions

$$\frac{\partial \Psi}{\partial R} = \frac{\Psi}{R} \tag{23}$$

and

$$\Psi(x,\ y) = \Psi_\delta(x). \tag{24}$$

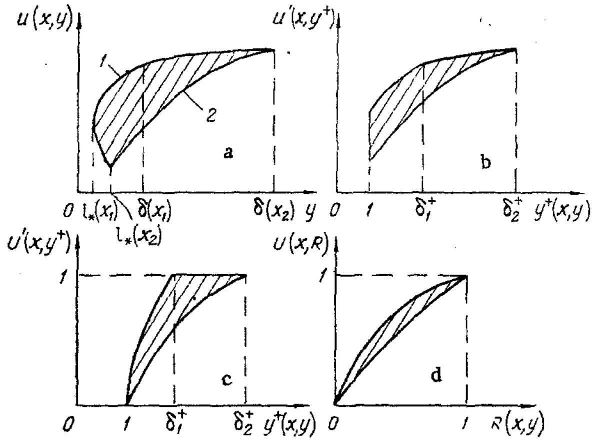

Fig. 1. Normalization of function u(x, t) with region of definition in Eq. (15): a) $u(x_1, y)$ (curve 1); $u(x_2, y)$ (curve 2).

Considering Eqs. (23) and (21) jointly, it follows that

$$U = 1, \tag{25}$$

since the segment $[R_1, R_2]$ contains the point $R_2 = 1$, for which $U = 1$.

Equation (23) and (25) hold outside the boundary layer, where $u(x, y) = u_\delta(x)$.

Considering Eqs. (24) and (21) jointly, it follows that

$$\frac{\partial U}{\partial R} = 1. \tag{26}$$

Since the segment $[R_1, R_2]$ contains the point $R_2 = 1$, the integral of Eq. (26) will be the equation

$$U = R.$$

As is evident from the results of [1], $R_1 \gtrsim \ln \delta_0^+/\ln \delta^+$ ($R_1 = \ln \delta_0^+/\ln \delta^+$ if there is no Karman transition region in the turbulent boundary layer).

Thus, for the case of a turbulent boundary layer, Eqs. (26), (27), and (9) are generalized analogs of Eqs. (2), (3), and (4) corresponding to a laminar boundary layer (sublayer).

The condition under which Eq. (9) transforms to Eq. (4) when $y^+ \to 1$ will not be determined.

From Eq. (24) both a universal logarithmic distribution law of the mean longitudnal velocity

$$u^+ = \frac{1}{\Psi_\delta} \ln y^+ + 1, \tag{28}$$

and a universal velocity-defect law

$$\frac{u_\delta - u}{u_*} = \frac{1}{\Psi_\delta} \ln\left(\frac{\delta}{y}\right)$$

are obtained. For small y^+, after series expansion of $\ln y^+$, it follows from Eq. (28) that $\Psi_\delta(x) = \Psi(x, y) = (y^+ - 1)/(u^+ - 1)$. Since $u^+ \to 1$ as $y^+ \to 1$, the function Ψ_* has an indeterminacy of 0/0 type at $y^+ = 1$. Removing this indeterminacy, it is found that $\Psi_* = (\partial u^+/\partial y^+)_*^{-1}$.

Hence, when

$$\Psi_* = 1 \tag{29}$$

Eq. (9) transforms to Eq. (4) when $y^+ \to 1$.

Note that the function Ψ obtained may be chosen as the turbulence criterion.

With increase in y^+, Ψ decreases in the laminar sublayer from 1 at the wall in accordance with the formula

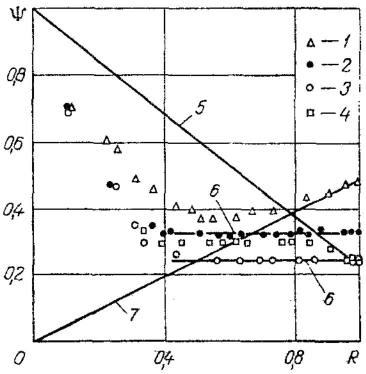

Fig. 2. Distribution of Ψ for various $K = (\nu/u_\delta^2)(\partial u_\delta/\partial x)$: 1) $K = 3.25 \cdot 10^{-6}$; 2) 0; 3) 0; 4) $1.05 \cdot 10^{-6}$; 5) according to Eq. (32); 6) Eq. (24); 7) Eq. (31).

$$\Psi = \frac{\ln y^+}{y^+ - 1} \tag{30}$$

to a minimum value corresponding to the onset of Kline pole disruption [2]. With further increase in y^+, Ψ rises insignificantly if there is a Karman transition region in the layer. Further, with a zero pressure gradient in the turbulent core and the outer part of the layer, Ψ remains constant and equal to Ψ_δ.

In the case of turbulent flow in a channel, in a tube, and at a plate, with $y^+ = \delta^+$ and a negative pressure gradient

$$\left(\frac{\partial \Psi}{\partial R}\right)_\delta = \Psi_\delta, \tag{31}$$

which follows from Eq. (23).

For a positive pressure gradient

$$\left(\frac{\partial \Psi}{\partial R}\right)_\delta = \Psi_\delta - 1, \tag{32}$$

and for a zero gradient

$$\left(\frac{\partial \Psi}{\partial R}\right)_\delta = 0. \tag{33}$$

The condition in Eq. (33) is obtained from Eq. (24). Because laminar mass penetrates into the turbulent layer for a smooth plate, with $y^+ \leq \delta^+$, $y^+/\delta^+ \sim 1$, Eq. (31) transforms smoothly to Eq. (33).

In Fig. 2, as an example, Eqs. (31)-(33) and (24) are compared with the experimental data of [2], where there was no Karman transition region in the turbulent boundary layer with zero pressure gradient. The experimental data are evidently in satisfactory agreement with the theoretical curves.

It may also be shown that $\Psi_\delta \to 1/3$ as $\delta^+ \to \infty$. Then Eq. (19) determines the friction at the wall, since, specifying ν, u_δ, and δ, u_* may be calculated by trial and error from Eq. (19).

For turbulent flow at a rough surface, $\Psi_* > 1$. Since $\Psi_\delta > 1$, $\Psi_s = 1$ is always found for $y^+ = y_s^+$ in view of the continuity of Ψ. Then the expression

$$u_s^+ = \ln y_s^+ + 1 \tag{34}$$

in the coordinates u^+, $\ln y^+$ determines the boundary surface along which fluctuate the turbulent eddies satisfying the fluctuation condition

$$R_s \frac{\partial U_s}{\partial R_s} = U_s, \tag{35}$$

where

$$U_s = \frac{u - u_s}{u_\delta - u_s}; \tag{36}$$

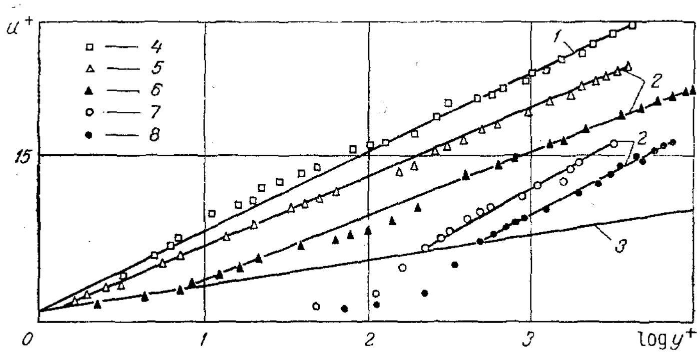

Fig. 3

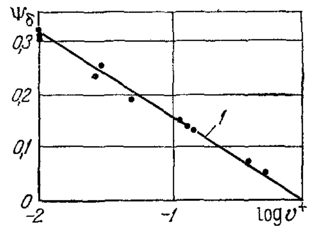

Fig. 4

Fig. 3. Distribution $u^+(y^+)$ for different dimensionless roughness heights k^+: 1) accord to Eq. (28); 2) Eq. (38); 3) Eq. (34); 4) $k^+ = 0$; 5) 19.2; 6) 68.5; 7) 119; 8) 268.

Fig. 4. Dependence of Ψ_δ on injection parameter v^+: 1) from Eq. (40).

$$R_s = \frac{\ln (y/y_s)}{\ln (\delta/y_s)} \tag{37}$$

and

$$U_s = R_s. \tag{38}$$

As an example, in Fig. 3, Eqs. (34), (38), and (28) are compared with the experimental data of [3]. The experimental data evidently agree satisfactorily with the theoretical curves.

It is evident from Fig. 3 that the velocity profile has an "elbow" (bend) at boundary surface 3, defined by Eq. (34); with increase in dimensionless roughness height $k^+ = k/l_*$ the dimensionless distance $y_s^+ = y_s/l_*$ to the surface 3 increases.

This is because fluctuation of the eddies is hindered in the region $y^+ < y_s^+$ by the presence of roughness of height k, whereas on surface 3 the mean turbulent eddies fluctuates as if at a smooth wall. Therefore, surface 3 is expediently chosen as an equivalent smooth wall, and the dimensionless distance to this surface y_s^+ as the equivalent dimensionless roughness height.

If, for some reason, the initial value Ψ_{δ_0}, is changed to $\Psi_{\delta K}$, it follows from Eq. (19) that

$$\Psi_{\delta 0} - \Psi_{\delta K} = \ln f(K), \tag{39}$$

where K is some parameter describing the change in the boundary layer (e.g., the dimensionless pressure gradient).

Using Eq. (39), the appropriate calculational formulas for complex liquid flow conditions may be constructed. For example, it may be shown that for $10^{-2} \leq v^+ \leq 1$ the effect of porous injection on the turbulent boundary layer is described by the relation

$$\Psi_{\delta v} = -\frac{1}{2} \Psi_{\delta 0} \log v^+. \tag{40}$$

In Fig. 4, Eq. (40) is compared with the data of [4]. The correlation between Eq. (40) and the experimental data of [4] is satisfactory, It is evident from Fig. 4 that if the injection velocity v approaches the dynamic velocity $u_*(v^+ = v/u_* \to 1)$ then the turbulent boundary-layer criterion $\Psi_{\delta v}$ is degenerate ($\Psi_{\delta v} \to 0$). This is because the turbulent boundary layer, in its usual physical meaning, ceases to exist when $v^+ \to 1$.

Thus, generalization of the Millionshchikov superposition method for turbulent eddies undulating along a solid wall has allowed some unknown laws to be discovered, new quantitative relations to be obtained, and calculations for complex flow conditions of the turbulent boundary layer to be simplified.

NOTATION

u, mean longitudinal velocity, m/sec; ν, kinematic viscosity, m^2/sec; $u_* = \sqrt{\tau_W/\rho}$, dynamic velocity, m/sec; ρ, density, kg/m^3; τ_W, tangential stress at wall, N/m^2; δ, boundary-layer thickness, m; $l_* = \nu/u_*$, transverse dimension of mean eddy at wall, m; $y^+ = y/l_*$, dimensionless coordinate; $u^+ = u/u_*$, dimensionless velocity; v, injection velocity, m/sec; $v^+ = v/u_*$, dimensionless injection parameter; k, roughness height, m. Subscripts: *, flow parameters for $y^+ = 1$; δ, flow parameters for $y = \delta$; W, wall parameter; s, flow parameter at rough surface with $\Psi = 1$; 0, initial flow conditions; v, flow conditions corresponding to injection velocity v.

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HYDRODYNAMIC REACTION TO ACCELERATED ONE-DIMENSIONAL MOTION OF GAS BUBBLES OF VARIABLE VOLUME IN AN UNBOUNDED LIQUID

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A solution is given for the problem of the hydrodynamic reaction of an ellipsoidal gas bubble of variable volume to accelerated motion and the relation between the value of the apparent mass and the eccentricity of the bubble.

In [1] the problem of the motion of gas bubbles with constant velocity in an unbounded volume of liquid was solved, and a relation was established between the constant velocity at which the bubble rises to the surface, the form of the bubble, and the forces of viscous friction. In a number of technical devices, motion with variable velocity and variable bubble volume is realized. The variability of the velocity leads to additional resisting forces [2]; according to the available literature, the magnitude of these forces as applied to the variable volume acting on a gas bubble in the form of an oblate ellipsoid of revolution has not been determined.

In an ellipsoidal coordinate system placed with center of mass of the bubble floating with velocity U,

$$x = c\,\mathrm{ch}\,\xi\cos\eta\cos\psi;\ \ y = c\,\mathrm{ch}\,\xi\cos\eta\sin\psi;\ \ z = c\,\mathrm{sh}\,\xi\sin\eta \tag{1}$$

for nonsymmetric potential motion the general solution for the velocity potential has the form [1]

$$\varphi(\xi,\ \eta) = [Ai\,\mathrm{sh}\,\xi - B(\mathrm{sh}\,\xi\,\mathrm{arc\,tg\,sh}\,\xi + 1)]\sin\eta. \tag{2}$$

Here $i = \sqrt{-1}$, and the unknown coefficients A and B are determined from the boundary conditions

$$\lim_{\xi\to\infty}\frac{1}{c\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}}\ \frac{\partial\varphi(\xi,\ \eta)}{\partial\xi} = U\sin\eta, \tag{3}$$

$$\left.\frac{\partial\varphi(\xi,\ \eta)}{\partial\vec{n}}\right|_{\xi_0} = c\ \frac{\partial\xi_0}{\partial t}\sqrt{\mathrm{ch}^2\xi_0 - \cos^2\eta}. \tag{4}$$

In Eq. (4) and below, the index 0 denotes points that belong to the surface of the bubble.

If we take into account (3) and (4), the expression for the velocity potential acquires the form

$$\varphi(\xi,\ \eta) = \frac{Uc}{L}\left[\left(L - \frac{\pi}{2}\right)\mathrm{sh}\,\xi + \mathrm{sh}\,\xi\,\mathrm{arc\,tg\,sh}\,\xi + 1\right]\sin\eta + \frac{c^2}{L}\ \frac{\partial\xi_0}{\partial t}\ \frac{\mathrm{ch}^2\xi_0 - \cos^2\eta}{\mathrm{ch}\,\xi_0}\left[\frac{\pi}{2}\,\mathrm{sh}\,\xi - \mathrm{sh}\,\xi\,\mathrm{arc\,tg\,sh}\,\xi - 1\right] \tag{5}$$

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